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Upper Semicontinuity of Pullback Attractors for a Nonautonomous Incompressible Non-Newtonian Fluid

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Abstract: In this paper, we study the pullback asymptotic behavior of solutions for a nonautonomous incompressible non-Newtonian fluid in two-dimensional bounded domains. We prove that our system possesses pullback attractors in the space H and then obtain upper semicontinuity (in δ) of pullback attractors of the nonautonomous incompressible non-Newtonian fluid with the nonautonomous perturbation under some proper assumptions, i.e., the pullback attractors $\mathcal{A}_\delta = \{A(t)\}_{t \in \mathbb{R}}$ of the incompressible non-Newtonian fluid with $\delta > 0$ and the global attractor A of (1) with $\delta = 0$ for any $t \in \mathbb{R}$.

Keywords: Incompressible non-Newtonian fluid, pullback attractors, pullback $D - \kappa$ contracting, upper semicontinuity

1 INTRODUCTION

In this paper, we shall discuss the upper semicontinuity of pullback attractors for the following non-autonomous incompressible non-Newtonian fluid in two-dimensional bounded domains

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p = \nabla \cdot \tau(e(u)) + \delta f(x, t), & x \in (x_1, x_2) \in \Omega, \\ \nabla \cdot u = 0, & x = (x_1, x_2) \in \Omega, \end{cases} \quad (1)$$

where $\Omega \in \mathbb{R}^2$ is a smooth bounded domain, $u = (u_1, u_2)$ denotes the velocity of the fluid, $f(x, t) = (f_1(x, t), f_2(x, t))$ is the time-dependent external force function, and p is the pressure. Equation (1) describes the motion of an isothermal incompressible viscous fluid, where $\tau(e(u)) = (\tau_{ij}(e(u)))_{2 \times 2}$ in which

$$\begin{aligned} \tau_{ij}(e(u)) &= 2\mu_0(\varepsilon + |e|^2)^{\frac{\alpha}{2}} e_{ij} - 2\mu_1 \Delta e_{ij}, \quad i, j = 1, 2, \\ e_{ij}(e(u)) &= e_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad |e|^2 = \sum_{i,j=1}^2 |e_{i,j}|^2, \quad (2) \\ \tau_{i,j,k} &= 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}, \quad k = 1, 2, \end{aligned}$$

and $\mu_0, \mu_1, \alpha, \varepsilon$ are parameters which generally

depend on the temperature and pressure. Here we assume that $\mu_0, \mu_1, \alpha, \varepsilon$ are positive constants and $0 < \alpha < 1$. In [4], Equation (1) is a Newtonian fluid if $\tau_{ij}(e(u))$ depends linearly on $e_{ij}(u)$. In our paper, we assume that the relation between $\tau_{ij}(e(u))$ and $e_{ij}(u)$ is nonlinear, which we called the fluid is non-Newtonian.

There are many works on the existence and uniqueness, regularity and long time behavior of weak solutions of (1) (see, e.g., [4, 5, 13, 15, 16]). In 1998, Nečasová and Lukáčov [16] obtained the existence and uniqueness of a solution to isothermal non-Newtonian bipolar fluid. In 2006, Nečasová and Penel [17] studied the asymptotic behavior in time of incompressible non-Newtonian fluids in the whole space assuming that initial data also belong to L^1 . As to the attractor, Bloom and Hao [6] proved the existence of the global attractor under the assumption that f belongs to certain weighted space. Later on, Li and Zhao [13] proved the existence of global attractor in the space H (see below for the definition of H), and they avoided using the weighted space. Zhao and Zhou [26] discussed the pullback asymptotic behavior of solutions for a non-autonomous incompressible

non-Newtonian fluid in two-dimensional bounded domains. Zhao, Li and Zhou [27] investigated the regularity of trajectory attractor and the upper semicontinuity in the spatial domains. In our paper, we adopt the idea of decomposition which is different from [26] and obtain the existence of pullback attractors for the equation (1). Another result in our paper is the upper semicontinuity in δ (not in the spatial domains in [26]) of pullback attractors for the equation (1).

Caraballo and Langa [7] gave a theorem on upper semicontinuity for pullback attractors (for random attractors, see, e.g., [8]), which was applied to prove the upper semicontinuity of pullback attractors for nonautonomous Navier-Stokes equations and reaction-diffusion equations. Wang and Qin [22] proved the upper semicontinuity of pullback attractors for nonclassical diffusion equations. In this paper, we shall study the upper semicontinuity of pullback attractors generated by solutions of equation (1) by applying the theory presented in [7], [8] and [22]. The main result of this paper can be stated as follows.

Theorem 1.1 Let $f_0 \in L^2_n(R;H)$ (see below for the definition of $f_0 \in L^2_n(R;H)$), then for any $f_0 \in H_0(f_0)$ and $u_0 \in H$ the pullback attractors $A_\delta = \{A_\delta(t)\}_{t \in R}$ for (1) with $\delta > 0$ and the global attractor A for (1) with $\delta = 0$ satisfy

$$\lim_{\delta \rightarrow 0^+} \text{dist}_H(A_\delta(t), A) = 0, \text{ for any } t \in R,$$

where dist_H is the distance in the norm topology of H .

The rest of the paper is organized as follows. In Section 2, we present existence results of solutions. In Section 3, we recall some theories of pullback attractor for general nonautonomous dynamical systems and give a technical method for verifying the upper semicontinuity of pullback attractors. In Section 4, we prove the upper semicontinuity of pullback attractors for equation (1) with $\delta > 0$.

2 PRELIMINARES

This section is devoted to present some abstract results on pullback attractors and their upper semicontinuity. Firstly, let us introduce the following notations:

$$\mathcal{U} = \{u \in (C_0^\infty(\Omega))^2, \text{div}u = 0\},$$

$$H = \text{closure of } \mathcal{U} \text{ in } (L^2(\Omega))^2 \text{ with norm } \|\cdot\|,$$

$$H' = \text{dual space of } H,$$

$$V = \text{closure of } \mathcal{U} \text{ in } (H^2(\Omega))^2 \text{ with norm } \|\cdot\|_V;$$

$$V' = \text{dual space of } V,$$

$$\langle \cdot, \cdot \rangle - \text{the inner product in } H,$$

$$\langle \cdot, \cdot \rangle - \text{the dual pairing between } V \text{ and } V',$$

$$\mathcal{B}(H) - \text{the union of all bounded sets of } H,$$

$$H_0(f) = \{f(\cdot, +s), s \in \mathbb{R}\} \text{ if } f(x, t) \in L^2_{loc}(\mathbb{R}; H),$$

$$L^2_b(\mathbb{R}; H) - \text{the set of functions } f \in L^2_{loc}(\mathbb{R}; H) -$$

Satisfying

$$\|f\|_{L^2_b(\mathbb{R}; H)}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|_H^2 ds < +\infty, \quad (3)$$

(a function f satisfies (3) is said to be translation bounded in $L^2_{loc}(\mathbb{R}; V')$).

Lemma 2.1 There exist two positive constants c_1 and c_2 which depend only on Ω such that

$$c_1 \|u\|_V^2 \leq a(u, u) \leq c_2 \|u\|_V^2, \quad \forall u \in V, \quad (4)$$

$$\text{where } a(u, v) = \sum_{i,j,k=1}^2 \left(\frac{\partial e_{i,j}(u)}{\partial x_k}, \frac{\partial e_{i,j}(v)}{\partial x_k} \right).$$

Proof. See, e.g., Bloom and Hao [5].

We see that $a(\cdot, \cdot)$ defines a positively definitely symmetric bilinear form on V and obtain an isometric operator $A \in \mathcal{L}(V, V')$, by $\langle Au, v \rangle = a(u, v)$, $\forall u, v \in V$, where $\langle \cdot, \cdot \rangle$ is the dual product between V' and V .

Let $A = P\Delta^2$, where P is the Leray projector from $L^2(\Omega)$ to H . We define $D(A) = \{u \in V : Au \in H\}$. Then $D(A)$ is a Hilbert space and A is also an isometry map from $D(A)$ to H . When $u \in D(A)$, we define $N(u)$ as

$$\langle N(u), v \rangle = - \int_{\Omega} \{\nabla \cdot [\mu(u)e(u)]\} \cdot v dx, \quad v \in H. \quad (5)$$

Then $N(u)$ is continuous from H to H .

Considering physics, the initial boundary value problem (1)-(2) can be formulated as follows:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot (2\mu_0(\varepsilon + |e|^2)^{-\alpha/2} e - 2\mu_1 \Delta e) + \delta f(x, t), & x \in \Omega, \\ \nabla \cdot u = 0, & x \in \Omega, \\ u = 0, \quad \tau_{ij} n_j n_i = 0, & x \in \partial\Omega, \\ u|_{t=\tau} = u_\tau, \quad \tau \in \mathbb{R}, \end{cases} \quad (6)$$

where $\tau_{ij} = 2\mu_l \frac{\partial e_{ij}}{\partial x_l}$ ($i, j, l = 1, 2$) and $n = (n_1, n_2)$ denotes

the exterior unit normal to the boundary $\partial\Omega$. The equality $u = 0$ in (6) is the no-slip boundary condition and the equality $\tau_{ij} n_j n_i = 0$ expresses the first moments of the traction vanish on $\partial\Omega$.

From the above analysis and excluding the pressure p , we express the weak version of problem (1) in the solenoidal vector fields as follow

$$\begin{cases} u_t + 2\mu_1 Au + B(u) + N(u) = \delta f(x, t), \\ u|_{t=\tau} = u_\tau, \end{cases} \quad (7)$$

where $B(u) = b(u, u) = (u \cdot \nabla)u$. For the details, we can refer to [5, 25].

Let X be a general Banach space with norm $\|\cdot\|_X$ and metric $d_X(\cdot, \cdot)$. A two-parameter family of mappings $\{U(t, \tau)\}_{t \geq \tau}$ is said to be a process in X if for $\forall t \geq s \geq \tau, \tau \in \mathbb{R}$, $U(t, s)U(s, \tau) = U(t, \tau)$ and $U(\tau, \tau) = Id$. Moreover, throughout the paper, we always assume that the process $U(\cdot, \cdot)$ is continuous in X , i.e., for any $t \geq \tau$, the mapping $U(t, \tau): X \rightarrow X$ is continuous. We denote by $dist_X(B_1, B_2)$, the Hausdorff semi-distance in X between $B_1 \subseteq X$ and $B_2 \subseteq X$, i.e., $dist_X(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} d_X(x, y)$, for $B_1, B_2 \subset X$,

where $d_X(x, y)$ denotes the distance between two points X and Y in X .

We now recall briefly some results on the theory of pullback attractors as developed in [9] and [11] by using the framework of evolution processes. After that, we formulate an abstract result in order to establish the existence of a pullback attractor for the nonautonomous dynamical system associated with equation (1).

Definition 2.1 A family of compact sets $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is said to be a pullback attractor if it satisfies the following:

- (i) (Invariance) $U(t, \tau)A(\tau) = A(t)$ for all $t \geq \tau$.
- (ii) (Attractivity) $\lim_{\tau \rightarrow +\infty} dist_X(U(t, \tau)B, A(t)) = 0$ for all bounded subset $B \subset X$.

Definition 2.2 The family of subsets $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ is said to be pullback absorbing with respect to the process $U(\cdot, \cdot)$ if, for every $t \in \mathbb{R}$ and any bounded subset $B \subset X$, there exists a time $T(t, B) > 0$ which depends on t and the bounded subset B , such that $U(t, t - \tau)B \subset D(t)$ for all $\tau \geq T(t, B)$.

Definition 2.3 Let $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ be a family of sets in X . A process $U(\cdot, \cdot)$ is said to be pullback \mathcal{D} -asymptotically compact in X if for any $t \in \mathbb{R}$, any sequences $\tau_n \rightarrow +\infty$ and $x_n \in D(t - \tau_n)$ the sequence $\{U(t, t - \tau_n)x_n\}$ is precompact in X .

The next theorem provide us a way to obtain the existence of pullback \mathcal{D} -attractors.

Lemma 2.2 Let the family of sets $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ be pullback absorbing for the process $U(\cdot, \cdot)$ and $U(\cdot, \cdot)$ is pullback \mathcal{D} -asymptotically compact in X . Then the family $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ defined by $A(t) = \Lambda(\mathcal{D}, t)$

where $\Lambda(\mathcal{D}, t) = \bigcap_{s \geq 0} \overline{\bigcup_{\tau \geq s} U(t, t - \tau)D(t - \tau)}$ for each $t \in \mathbb{R}$ is a pullback attractor for $U(\cdot, \cdot)$ in X .

Lemma 2.2 can be seen as a consequence of Theorem 7 in [9], and we refer its proof to [3] and [21]. To get the pullback \mathcal{D} -asymptotically compact in the above theorem, we can use the properties Kuratowski measure of noncompactness. So we have to introduce the concept of noncompactness measure. The details can be found in [18].

Definition 2.4 ([18]) Let $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ be a family of sets in X . A process $U(\cdot, \cdot)$ is said to be pullback \mathcal{D} - κ contracting, if for any $t \in \mathbb{R}$, $\sigma > 0$, there exists a time $T_D(t, \sigma) > 0$, such that $\kappa(U(t, t - \tau)D(t - \tau)) \leq \sigma$, for all $\tau \geq T_D(t, \sigma)$, where $\kappa(B)$ is the Kuratowski measure of noncompactness.

Lemma 2.3

Let $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$, $\widehat{\mathcal{D}} = \{\widehat{D}(t)\}_{t \in \mathbb{R}}$ be two families of sets in X , and satisfy that for any $t \in \mathbb{R}$, there exists a time $T_{\mathcal{D}, \widehat{\mathcal{D}}}(t) > 0$ such that $U(t, t - \tau)D(t - \tau) \subset \widehat{D}(t)$ for all $\tau \geq T_{\mathcal{D}, \widehat{\mathcal{D}}}(t)$. If it is pullback $\widehat{\mathcal{D}}$ - κ contracting, then $U(\cdot, \cdot)$ is pullback \mathcal{D} -asymptotically compact.

Proof. See, e.g., Wang and Qin [22].

Theorem 2.1 Assume that the assumptions in Lemma 2.3 hold. If the process $U(\cdot, \cdot)$ is pullback $\widehat{\mathcal{D}}$ - κ contracting and the family of sets $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ is pullback absorbing for $U(\cdot, \cdot)$, then $U(\cdot, \cdot)$ has a pullback attractor.

From Lemma 2.3 and the property of the Kuratowski measure of noncompactness that $\kappa(D) = 0$ if and only if \overline{D} is compact, where \overline{D} is the closure of D , and the conclusion follows easily.

According to Theorem 2.1, pullback \mathcal{D} - κ contracting is important to prove our existence of pullback attractors. So we give the following lemma first.

Lemma 2.4 Let $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ be a family of sets in X .

Suppose $U(\cdot, \cdot) = U_1(\cdot, \cdot) + U_2(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times X \rightarrow X$, such that

(i) for any $t \in \mathbb{R}$, $\|U_1(t, t-\tau)x_{t-\tau}\|_X \leq \Phi(t, \tau)$

for all $x_{t-\tau} \in D(t-\tau)$, $\tau > 0$,

where $\Phi(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies $\lim_{\tau \rightarrow +\infty} \Phi(t, \tau) = 0$,

for each $t \in \mathbb{R}$.

(ii) for any $t \in \mathbb{R}$ and any $T \geq 0$, $\cup_{0 \leq \tau \leq T} U_2(t, t-\tau)D(t-\tau)$ is bounded and $U_2(t, t-\tau)D(t-\tau)$ is precompact in X for any $\tau > 0$. Then the process $U(\cdot, \cdot)$ is pullback $\mathcal{D} - \kappa$ contracting in X .

Proof. See, e.g., Wang and Qin[22].

Remark The idea that assumptions (i) and (ii) in Lemma 2.4 can be shown by decomposing solutions of (1) has been widely applied to dissipative dynamical systems ([1, 12, 19, 23, 24]).

3 NONAUTONOMOUS PERTURBATION AUTONOMOUS SYSTEMS

We shall consider the relationship between the pullback attractors $\mathcal{A}_\delta = \{A_\delta(t)\}_{t \in \mathbb{R}}$ for the perturbed nonautonomous system with $\delta > 0$ and the global attractor A for the unperturbed autonomous system with $\delta = 0$ in this section. We shall show that for any $t \in \mathbb{R}$, $A_\delta(t)$ lies within a small neighborhood of A , and this implies that, when δ is bounded, $A_\delta(t)$ will not "blow up".

By Lemma 3.1 (i) in [26], there is a C_0 -semigroup defined on X , that is, A is a compact set, invariant and global attracting every bounded sets in X ($\lim_{t \rightarrow +\infty} dist_X(S(t)B, A) = 0$ for all bounded subset $B \subset X$). Various conditions for the existence of global attractors can be seen in references [2, 12, 14, 20], and references there in.

We now perturb the semigroup defined above by a nonautonomous term on a small parameter $\delta \in (0, \delta_0]$, thus we obtain a nonautonomous dynamical system driven by a process $U_\delta(\cdot, \cdot)$. We also make the following assumption.

(H₁) For each $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, and $x \in X$, we have

$$\lim_{\delta \rightarrow 0} d_X(U_\delta(t, t-\tau)x, S(\tau)x) = 0,$$

uniformly on bounded sets of X . Then from Caraballo *et al* [8], we have next theorem.

Lemma 3.1 Assume that (H₁) holds, and for any $\delta \in (0, \delta_0]$, there exists a pullback attractor $\mathcal{A}_\delta = \{A_\delta(t)\}_{t \in \mathbb{R}}$. If there exists a compact set $K \subset X$ such that

$$(H_2) \lim_{\delta \rightarrow 0} dist_X(A_\delta(t), K) = 0 \text{ for any } t \in \mathbb{R}.$$

Then, \mathcal{A}_δ and A have the upper semicontinuity, that is,

$$\lim_{\delta \rightarrow 0} dist_X(A_\delta(t), A) = 0 \text{ for any } t \in \mathbb{R}.$$

Our aim is to apply Lemma 3.1 to obtain the upper semicontinuity of pullback attractors \mathcal{A}_δ . We now present a technical method for verifying (H₂) for the process generated by (1) with $\delta > 0$.

Lemma 3.2 Assume that the family $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ is pullback absorbing for $U(\cdot, \cdot)$, and for each $\delta \in (0, \delta_0]$, $\mathcal{K}_\delta = \{K_\delta(t)\}_{t \in \mathbb{R}}$ is a family of compact sets in X .

Suppose $U_\delta(\cdot, \cdot) = U_{1,\delta}(\cdot, \cdot) + U_{2,\delta}(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \times X \rightarrow X$,

such that

(i) for any $t \in \mathbb{R}$, and any $\delta \in (0, \delta_0]$

$$\|U_{1,\delta}(t, t-\tau)x_{t-\tau}\|_X \leq \Phi(t, \tau), \text{ for } x_{t-\tau} \in D(t-\tau), \tau > 0,$$

where $\Phi(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies $\lim_{\tau \rightarrow +\infty} \Phi(t, \tau) = 0$

for each $t \in \mathbb{R}$;

(ii) for any $t \in \mathbb{R}$ and any $T \geq 0$, $\cup_{0 \leq \tau \leq T} U_{2,\delta}(t, t-\tau)D(t-\tau)$ is bounded, and for any $T \geq 0$ there exists a time $T_D(t) > 0$, which is independent of δ such that $U_{2,\delta}(t, t-\tau)D(t-\tau) \subset K_\delta(t)$, for $\tau \geq T_D(t)$, $\delta \in (0, \delta_0]$ and there exists a compact set $K \subset X$ such that (H₂) $\lim_{\delta \rightarrow 0} dist_X(K_\delta(t), K) = 0$ for any $t \in \mathbb{R}$.

Then for each $\delta \in (0, \delta_0]$, there exists a pullback attractor $\mathcal{A}_\delta = \{A_\delta(t)\}_{t \in \mathbb{R}}$, and (H₂) holds.

Proof. See, e.g., Wang and Qin [22].

4 THE PROOF OF UPPER SEMICONTINUITY OF PULLBACK ATTRACTORS

In this section, we shall prove the upper semicontinuity for the pullback attractors generated by (1) with $\delta > 0$ and the global attractor generated by equation (1) with $\delta = 0$. We use a new decomposition technique to obtain that there exists a pullback attractor for the process. Using Lemma 3.2 and Lemma 3.1, our main result of this paper can be obtained.

Lemma 4.1 Assume $\mu_0, \mu_1, \varepsilon > 0$ and $\alpha \in (0, 1)$.

For any $t \in \mathbb{R}$, the group $S(t)$ generated by equation (1) when $\delta = 0$ possesses a global attractor A .

Proof. See in [13].

Definition 4.1 A function $f \in L^2_n(\mathbb{R}; H)$ if for any $\epsilon > 0$, there exists a constant $\sigma = \sigma(\epsilon) > 0$ such that $\sup_{t \in \mathbb{R}} \int_t^{t+\sigma} \|f(s)\|^2 ds \leq \epsilon$. Clearly, there holds

$$L^2_n(\mathbb{R}; H) \subseteq L^2_b(\mathbb{R}; H).$$

Lemma 4.2 If $f_0 \in L^2_0(\mathbb{R}; H)$, then for any $f \in H_0(f_0)$ and any $u_0 \in H$, problem (2.5) admits a unique solution u satisfying

$$u \in C(\mathbb{R}; H) \cap L^\infty(\mathbb{R}; H) \cap L^2_{loc}(\mathbb{R}; V), \partial_t u \in L^2_{loc}(\mathbb{R}; V').$$

Proof. See, e.g., [25].

From Lemma 4.2, for each $\delta > 0$, the solution operator $U_\delta(t, \tau)$ ($t > \tau$) of problem (6) forms a process on H . For more details, we can see in [10]. So, we decompose the solution $u^\delta(t) = U_\delta(t, \tau)u_\tau$ of (7) with initial data $u_\tau \in H$ as follows

$$u^\delta = U_\delta(t, \tau)u_\tau = U_{1,\delta}(t, \tau)u_\tau + U_{2,\delta}(t, \tau)u_\tau \quad (8)$$

Where

$$U_{1,\delta}(t, \tau)u_\tau = v(t), \quad (9)$$

$$U_{2,\delta}(t, \tau)u_\tau = w(t), \quad (10)$$

solve the following problems, respectively

$$\begin{cases} v_t + 2\mu_1 Av = 0, & \text{in } (x, t) \in \Omega \times [\tau, +\infty), \\ v(x, t) = 0, & \text{on } \partial\Omega \times [\tau, +\infty), \\ v(\tau, x) = u_\tau, & x \in \Omega, \end{cases} \quad (11)$$

and

$$\begin{cases} w_t + 2\mu_1 Aw + B(u) + N(u) = \delta f(x, t), & (x, t) \in \Omega \times [\tau, +\infty), \\ w(x, t) = 0, & \text{on } \partial\Omega \times [\tau, +\infty), \\ w(\tau, x) = 0, & x \in \Omega. \end{cases} \quad (12)$$

Lemma 4.3 For any bounded set $B \subset H$ and any $t \in \mathbb{R}$, there exists a time $T(B, t) > 0$ which depend on B and t , such that $\|U_\delta(t, t-\tau)u_{t-\tau}\|^2 \leq R_\delta(t)$ for all $\tau \geq T(B, t)$ and any $u_{t-\tau} \in B$, where $R_\delta(t) = C(\delta^2 e^{-c_1\mu_1 t} \int_{-\infty}^t e^{c_1\mu_1 s} \|f(s)\|_H^2 ds + 1)$, C is a positive constant independent of B, t, τ .

Proof. Multiplying problem (7) by u , applying Lemma 2.1 and the non-negativity of the term $\langle N(u), u \rangle$, and integrating the result over Ω , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + 2c_1\mu_1 \|u(t)\|_V^2 \\ & \leq \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + 2\mu_1 a(u(t), u(t)) + \langle N(u(t)), u(t) \rangle \\ & = (\delta f(t), u(t)) \\ & \leq \frac{\delta^2 \|f(t)\|^2}{6c_1\mu_1} + \frac{3c_1\mu_1}{2} \|u(t)\|^2. \end{aligned} \quad (13)$$

We know the fact that $\|u\|^2 \leq \|u\|_V^2$ holds for any $u \in V$ from the definition of norm of space V . Thus we obtain from (13)

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + 2c_1\mu_1 \|u(t)\|_V^2 \leq \frac{\delta^2 \|f\|^2}{6c_1\mu_1} + \frac{3c_1\mu_1}{2} \|u(t)\|_V^2.$$

$$\text{That is, } \frac{d}{dt} (e^{c_1\mu_1 t} \|u(t)\|^2) \leq e^{c_1\mu_1 t} \frac{\delta^2 \|f\|^2}{3c_1\mu_1}.$$

Integrating the above inequality from $t-\tau$ to t , we get

$$\|u(t)\|^2 \leq e^{-c_1\mu_1 \tau} \|u(t-\tau)\|^2 + \frac{\delta^2}{3c_1\mu_1} \int_{t-\tau}^t e^{c_1\mu_1(s-t)} \|f(s)\|^2 ds,$$

which implies that we can find a time $T(t, B) > 0$ large enough such that

$$\|u(t)\|^2 \leq \frac{2}{3c_1\mu_1} (\delta^2 e^{-c_1\mu_1 \tau} \int_{-\infty}^t \|f(s)\|^2 ds + 1) \quad (14)$$

for any $\tau > T(t, B)$ and any $u_{t-\tau} \in B$. We end the proof of Lemma 4.3. \square

If we set

$$B_\delta(t) = \{u(t) \in H, \|u(t)\|^2 \leq R_\delta(t)\},$$

where $R_\delta(t)$ defined in Lemma 4.3, it is easy to check that $\mathcal{B}_\delta = \{B_\delta(t)\}$ is pullback absorbing in H .

It also implies that, for any $\delta > 0$,

$$\lim_{t \rightarrow +\infty} e^{c_1\mu_1 t} R_\delta(t) = 0. \quad (15)$$

Lemma 4.4 Let $R_\delta(t), B_\delta(t)$ be given as above.

(i) For any $t \in \mathbb{R}$, the solution

$v(t) = U_{1,\delta}(t, t-\tau)u_{t-\tau}$ of (4.4) satisfies

$$\|U_{1,\delta}(t, t-\tau)u_{t-\tau}\|^2 \leq e^{-c_1\mu_1 \tau} R_\delta(t-\tau), \quad (16)$$

for all $\tau \geq 0$ and $u_{t-\tau} \in B_\delta(t-\tau)$.

(ii) There exists a function $\Phi(t, \tau)$ depending on t and τ such that $\lim_{\tau \rightarrow +\infty} \Phi(t, \tau) = 0$.

Proof. Multiplying (1) with v and integrating over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + 2\mu_1 a(v(t), v(t)) = 0.$$

Applying the Gronwall's inequality, we have $\|v(t)\|^2 \leq e^{-4c_1\mu_1 \tau} \|v(t-\tau)\|^2 \leq e^{-4c_1\mu_1 \tau} R_\delta(t)$.

We set $\Phi(t, \tau) = e^{-4c_1\mu_1 \tau} R_\delta(t)$, and get $\lim_{\tau \rightarrow +\infty} \Phi(t, \tau) = 0$.

Lemma 4.5 Let $\mathcal{B}_\delta(t) = \{B_\delta(t)\}_{t \in \mathbb{R}}$ be given above.

For any $t \in \mathbb{R}$, there exists a time $T_\delta(t, \mathcal{B}) > 0$ and $I_\delta(t) > 0$, the solution of Equation(11), $U_{2,\delta}(t, \tau)u_\tau = w(t)$ satisfies

$$\|U_{2,\delta}(t, t-\tau)u_{t-\tau}\|_V^2 \leq I_\delta(t) \quad (17)$$

for any $\tau \geq T_\delta(t, \mathcal{B})$ and any $u_{t-\tau} \in B_\delta(t-\tau)$.

Proof. Multiplying equation (11) with w_t and integrating over Ω , we have

$$\begin{aligned} & \|w_t\|^2 + c_1\mu_1 \frac{d}{dt} \|w\|_V^2 \leq \|w_t\|^2 + 2\mu_1 a(w, w_t) \\ & \leq |b(u, u, w_t)| + |\langle N(u), w_t \rangle| + |\langle \delta f(x, t), w_t \rangle|. \end{aligned} \quad (18)$$

At the same time, by the Hölder inequality and the Gagliardo-Nirenberg inequality, there exists a constant $c_3 > 0$ such that

$$\begin{aligned} b(u, u, w_t) &\leq \|u\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|w_t\| \\ &\leq c_3 \|u\|^{\frac{1}{2}} \|u\|_{H^1(\Omega)}^{\frac{1}{2}} \|u\|_{H^1(\Omega)}^{\frac{1}{2}} \|u\|_{V}^{\frac{1}{2}} \|w_t\| \\ &\leq c_3 \|u\|_{V}^2 \|w_t\|, \end{aligned} \quad (19)$$

Where we have used the fact that $\|u\|^{\frac{1}{2}} \leq \|u\|_{V}^{\frac{1}{2}}$ and $\|u\|_{H^1} \leq \|u\|_{V}$.

The estimate on the term $\langle N(u), w_t \rangle$ in (4.13) can be estimated as follows

$$\begin{aligned} \langle N(u), w_t \rangle &= 2 \int_{\Omega} \mu(e(u)) e_{i,j}(u) e_{i,j}(w_t) dx \\ &\leq \frac{2\mu_0}{\varepsilon^{\frac{\alpha}{2}}} \int_{\Omega} |e_{ij}(u) e_{ij}(w_t)| dx \\ &\leq c_4 \|u\|_{H_0^1} \|w_t\|_{H_0^1}, \end{aligned} \quad (20)$$

where we have used the fact that

$$\mu(u) = \mu_0 (\varepsilon + e_{ij} e_{ij})^{-\frac{\alpha}{2}} \leq \mu_0 \varepsilon^{-\frac{\alpha}{2}}.$$

Substituting (19) and (20) into (18), using the Poincar inequality, we have

$$\begin{aligned} &\|w_t\|^2 + c_1 \mu_1 \frac{d}{dt} \|w\|^2 \\ &\leq \langle \delta f, w_t \rangle + c_3 \|u\|_{V}^2 \|w_t\| + c_4 \|u\|_{H_0^1} \|w_t\|_{H_0^1} \\ &\leq \frac{3c_3^2}{2} \|u\|_{V}^4 + \frac{1}{3} \|w_t\|^2 + \frac{3c_5^2}{2} \|u\|_{H_0^1}^2 \\ &\quad + \frac{1}{3} \|w_t\|^2 + \frac{3\delta^2}{2} \|f\|^2 + \frac{1}{3} \|w_t\|^2 \end{aligned}$$

That is,

$$c_1 \mu_1 \frac{d}{dt} \|w\|_{V}^2 \leq \frac{3c_3^2}{2} \|u\|_{V}^4 + \frac{3c_5^2}{2} \|u\|_{H_0^1}^2 + \frac{3\delta^2}{2} \|f\|^2. \quad (21)$$

Integrating (21) from $t - \tau$ to t , we obtain

$$\begin{aligned} \|w\|_{V}^2 &\leq \frac{1}{c_1 \mu_1} \int_{t-\tau}^t \left(\frac{3c_3^2}{2} \|u(s)\|_{V}^4 + \frac{3c_5^2}{2} \|u(s)\|_{H_0^1}^2 \right. \\ &\quad \left. + \frac{3\delta^2}{2} \|f(s)\|^2 \right) ds + \|w_{t-\tau}\|_{V}^2 \equiv I_{\delta}(t), \end{aligned} \quad (22)$$

where here we have used the fact $\|u(t)\|_{H_0^1}^2 \leq \|u(t)\|_{V}^2$.

This finishes the proof.

Lemma 4.6 For any $t \in \mathbb{R}$, any $\tau > 0$ the solution $u_{\delta}(t) = U_{\delta}(t, \tau) u_{\tau}$ of Equation(6) with $\delta > 0$ converges in H as $\delta \rightarrow 0^+$ to the solution $u(t) = S(t) u_{\tau}$ of the unperturbed problem equation (6) with $\delta = 0$, uniformly when u_0 varies in bounded sets, that is,

$$\limsup_{\delta \rightarrow 0^+} \sup_{u_0 \in B} \|u_{\delta}(t) - u(t)\| = 0, \quad (23)$$

where B is a bounded subset in H .

Proof. Define $y^{\delta}(t) = u_{\delta}(t) - u(t)$, then clearly that $y^{\delta}(t)$ satisfies

$$\begin{aligned} \frac{d}{dt} y^{\delta} + 2\mu_1 a(y^{\delta}, y^{\delta}) + B(u^{\delta}) - B(u) + N(u^{\delta} - N(u)) \\ = \sigma f(x, t) \end{aligned} \quad (24)$$

Taking inner product (\cdot, \cdot) of (24) with y^{δ} and using Lemma 2.1, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y^{\delta}\|^2 + 2c_1 \mu_1 \|y^{\delta}\|_{V}^2 + \langle B(u^{\delta}) - B(u), y^{\delta} \rangle \\ + \langle N(u^{\delta}) - N(u), y^{\delta} \rangle \leq \langle \delta f(x, t), y^{\delta} \rangle \end{aligned} \quad (25)$$

From the monotonicity of $\mu(u)$, it follows that

$$\begin{aligned} \langle N(u^{\delta}) - N(u), y^{\delta} \rangle \\ = 2 \int_{\Omega} [\mu(u^{\delta}) e_{ij}(u^{\delta}) - \mu(u) e_{ij}(u)] e_{ij}(y^{\delta}) dx \geq 0. \end{aligned} \quad (26)$$

From the definition of the operator $B(u, u)$, we get

$$\begin{aligned} 2|\langle B(u^{\delta}) - B(u), y^{\delta} \rangle| &= |b(u^{\delta}, u^{\delta}, y^{\delta}) - b(u, u, y^{\delta})| \\ &= 2|b(u^{\delta}, u^{\delta}, y^{\delta}) - b(u, u^{\delta}, y^{\delta}) \\ &\quad + b(u, u^{\delta}, y^{\delta}) - b(u, u, y^{\delta})| \\ &= 2|b(y^{\delta}, u^{\delta}, y^{\delta}) - b(u, y^{\delta}, y^{\delta})| \\ &= 2|b(y^{\delta}, u^{\delta}, y^{\delta})| \\ &\leq 2\|y^{\delta}\|_{L^4} \|u\|_{L^4} \|y^{\delta}\| \\ &\leq c_3 \|y^{\delta}\|_{L^2}^{\frac{1}{2}} \|y^{\delta}\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|u\|_{V}^{\frac{1}{2}} \|y^{\delta}\| \\ &\leq c_3 \|y^{\delta}\|_{V} \|u\|_{V} \|y^{\delta}\|_{L^2}, \end{aligned} \quad (27)$$

where we have used the fact that $b(u, y^{\delta}, y^{\delta}) = 0$.

Combining (25)-(27), we obtain by using the Young inequality

$$\begin{aligned} \frac{d}{dt} \|y^{\delta}\|^2 + 4c_1 \mu_1 \|y^{\delta}\|_{V}^2 \\ \leq 2\langle B(u^{\delta}) - B(u), y^{\delta} \rangle - 2\langle N(u^{\delta}) - N(u), y^{\delta} \rangle + 2\langle \delta f, y^{\delta} \rangle \\ \leq c_3 \|y^{\delta}\|_{V} \|u\|_{V} \|y^{\delta}\|_{L^2} + 2\langle \delta f, y^{\delta} \rangle \\ \leq 2c_1 \mu_1 \|y^{\delta}\|_{V}^2 + \frac{c_3^2}{8c_1 \mu_1} \|u\|_{V}^2 \|y^{\delta}\|^2 \\ + 2c_1 \mu_1 \|y^{\delta}\|_{V}^2 + \frac{\delta^2}{2c_1 \mu_1} \|f\|^2. \end{aligned} \quad (28)$$

Therefore, we get

$$\frac{d}{dt} \|y^{\delta}\|^2 \leq \frac{\delta^2}{2c_1 \mu_1} \|f\|^2 + \frac{c_3^2}{8c_1 \mu_1} \|u\|_{V}^2 \|y^{\delta}\|^2.$$

Applying the Gronwall inequality, we obtain that there exist some positive constants c_5 and c_6 such that

$$\|y^{\delta}\|^2 \leq c_5 \delta^2 \left(\int_{t-\tau}^t \|f\|^2 ds \right) \exp \left(c_6 \int_{t-\tau}^t \|u\|_{V}^2 ds \right) \quad (29)$$

which implies (23).

Now we are ready to prove Theorem 1.1. In fact, combining Lemmas 4.3-4.6 with Lemma 3.1 and Lemma 3.2, we can obtain Theorem 1.1 immediately.

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References

- [1] Arrieta, J., Carvalho, A. N., and Hale, J. K., 1992. A damped hyperbolic equation with critical exponent, *Communications in partial differential equations*, no.17, pp.841-866.
- [2] Babin, A. V. and Vishik, M. I., 1992. *Attractors of evolution equations*. Elsevier, North-Holland, Amsterdam.
- [3] Bates, P. W., Lisei, H., and Lu, K., 2006. Attractors for stochastic lattice dynamical systems. *Stochastics Dyn.*, vol. 6, no.1, pp.1-22.
- [4] Bellout, H., Bloom, F., and Necas, J., 1993. Weak and measure-valued solutions for non-Newtonian fluids, *CR Acad. Sci. Paris*, no.317, pp. 795-800.
- [5] Bloom, F. and Hao, W., 2001. Regularization of a non-Newtonian system in an unbounded channel: Existence and uniqueness of solutions, *Nonlinear Anal., TMA*, vol. 44, no. 3, pp. 281-309.
- [6] Bloom, F. and Hao, W., 2001. Regularization of a non-Newtonian system in an unbounded channel: Existence and uniqueness of a maximal compact attractor, *Nonlinear Anal., TMA*, vol.43, pp. 743-766.
- [7] T. Caraballo and J. A. Langa, 2003. On the upper semicontinuity of attractors for nonautonomous and random dynamical systems, *Dyn. Contin. Discrete Impulsive Syst., Ser. A: Math. Anal.*, vol. 10, pp. 491-513.
- [8] T. Caraballo, J. A. Langa, and J. C. Robinson, 1998. Upper semicontinuity of attractors for small random perturbations of dynamical systems, *Comm. Partial Differential Equations*, vol. 23, pp. 1557-1581.
- [9] T. Caraballo, G. Lukaszewicz, and J., 2006. Real, Pullback attractors for asymptotically compact nonautonomous dynamical systems, *Nonlinear Anal., TMA*, vol. 64, pp. 484-498.
- [10] V. Chepyzhov and M. I. Vishik, 2002. *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence, RI, vol. 49.
- [11] H. Crauel and F. Flandoli, 1994. *Attractors for random dynamical systems*, *Probab. Theory Relat. Fields*, Vol. 100, pp. 365-393.
- [12] Hale, J. K. *Asymptotic Behavior of Dissipative Systems*, 1988. Amer. Math. Soc., Providence.
- [13] Y. Li and C. Zhao, 2002. Global attractor for a non-Newtonian system in two-dimensional unbounded domains, *Acta Anal. Funct. Appl.*, vol. 4, pp. 343-349.
- [14] Q. F. Ma, S. H. Wang, and C. K. Zhong, 2002. Necessary and sufficient conditions for the existence of global attractors for semigroups and applications, *Indiana Univ. Math. J.*, vol. 51, pp. 1541-1559.
- [15] Necas, J., Málek, J., Rokyta, M., and Ruzicka, M. 1996. *Weak and measure-valued solutions to evolutionary PDEs*, Vol. 13. CRC Press.
- [16] Matušič, S. and Medvid'ová-Lukáč, M., 1998. Bipolar isothermal non-Newtonian compressible fluids., *J. Math. Anal. Appl.*, vol. 225, 168-192.
- [17] Nečasová, Š. and Penel, P., 2006. Incompressible non-Newtonian fluids: Time asymptotic behaviour of weak solutions, *Math. Method. Appl. Sci*, vol. 14, no.29, pp. 1615-1630.
- [18] G. Sell and Y. You, 2002. *Dynamics of Evolutionary Equations*, Springer-Verlag, New York, vol. 143.
- [19] C. Y. Sun and M. H. Yang, 2008. Dynamics of the nonclassical diffusion equations, *Asymptotic Anal.*, vol. 59, pp. 51-81.
- [20] R. Temam, 1997. *Infinite-Dimensional dynamical Systems in Mechanics and Physics*, 2nd ed., Springer-Verlag, New York.
- [21] Y. H. Wang, 2008. Pullback attractors for nonautonomous wave equations with critical exponent, *Nonlinear Anal., TMA*, vol. 68, pp. 365-376.
- [22] Y. H. Wang and Y. Qin, 2010. Upper semicontinuity of pullback attractors for nonclassical diffusion equations, *J. Math. Phys.*, vol. 51, pp. 1-10.
- [23] Y. L. Xiao, 2002. Attractors for a nonclassical diffusion equation, *Acta Math. Sinica. English Ser.*, vol. 18, pp. 273-276.
- [24] S. Zelik, 2004. Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent, *Comm. Partial Differential Equations*, vol. 3, pp. 921-934.
- [25] C. Zhao and S. Li, 2004. H²-compact attractor for a non-Newtonian system in two dimensional unbounded domains, *Nonlinear Anal., TMA*, vol. 7, pp. 1093-1103.

- Nonlinear Anal., TMA, vol. 7, pp. 1093-1103.
- [26] C. Zhao and S. Zhou, 2007. Pullback attractors for a non-autonomous incompressible non-Newtonian fluid, *J. Differential Equation*, vol. 283, pp. 394-425.
- [27] C. Zhao, Y. Li and S. Zhou, 2009. Regularity of trajectory attractor and upper semicontinuity of global attractor for a 2D non-Newtonian fluid, *J. Differential Equations*, vol. 247, pp. 2331-2363.