A Constrained Cost Minimizing Redundancy Allocation Problem in Coherent Systems with Non-overlapping Subsystems

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Abstract: The present paper considers the allocation of redundancy to coherent systems with competing choices of system-components, while minimizing the total cost of adding redundancy, subject to a predetermined level of system reliability. Use of redundancy to enhance system reliability increases the total design cost. This paper resolves a cost minimizing redundancy allocation problem (CMRAP) in a coherent system, where total cost of using redundancy is minimized subject to a given reliability target. A numerical example has been included to explicate the method. A sensitivity analysis has been done to study the sensitivity of the optimal solution, related cost and the gain in system reliability to the reliability targets. It has been observed that the solution is robust within a group of reliability targets, but they are sensitive from one group to the other. No fixed form of component life distribution has been assumed here, which added enough flexibility in application of this method.

Keywords: Coherent system; Cost minimization; Redundancy; Reliability target; System reliability

1 INTRODUCTION

In this article an active redundancy is considered where the redundant components are connected in parallel to the original components of the system, and the original and the redundant component, both function simultaneously. When one fails, other continues to work so that the system continues to function without interruption. Usually, an active redundancy is used in case it is difficult, if not impossible, to replace the failed components when the system is in operation. By the property of a coherent system, system reliability can be enhanced by increasing the redundancy into the system. Here we consider coherent systems which can be decomposed into a number of non-overlapping subsystems such that the system fails with the failure of any of the subsystems. Thus it is essential to strengthen the subsystems, which can be done by using redundancy. But the amount of increase in system reliability varies with the number of redundant components added. The problem becomes complex when the optimal number of redundant components is to be decided that minimizes the total cost under some reliability constraints.

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Unconstrained redundancy allocation problem or reliability maximizing redundancy allocation problem under cost (or some other) constraints are widely discussed problems found in the literature. Mention may be made of the work of Morrison [1] which considered the optimal allocation of spares in systems with two subsystems in the problem of maximizing system life; Boland et al. [2] discussed a redundancy allocation problem for series and parallel systems; Shaked and Shanthikumar [3] studied the problem of allocating m active redundancies to an n-component series system where the lifetimes of the original components and redundant components are identically and independently distributed; Coit and Smith [4] considered reliability optimization of series-parallel systems using genetic algorithm; Ahmadizar and Soltanpanah [5] solved a reliability optimization problem for a series system under budget constraints. Often cost minimization becomes of more concern than reliability maximization to the reliability practitioners, ensuring a specified reliability level of the system. Note that the reliability of a system can be enhanced whenever redundant components are added to its components, but that would result in a costlier system. Therefore a trade-off is necessary while improving system reliability using redundancy. Ruan and Sun [6] solved a cost minimization problem for series systems. The present paper solves a cost minimization problem where redundancy allocation has been made under a reliability constraint. The cost of adding redundancy is so minimized that a reliability target must be met. The redundancy number for each subsystem that minimizes the total cost is the decision variable.

This paper considers a wide range of coherent systems, decomposable into a number of non-overlapping subsystems, which is commonly referred to as having the series-parallel structures and very much used in various fields of important applications, such as a coal transportation system in coal mines where coal is transported from bin to boiler through primary feeders connected in parallel, a reclaimer in series, secondary feeders in parallel, an air conditioning system in which number of air conditioning machines in parallel are connected to the power source in series, a garden sprinkler system with sensors in parallel, a controller and a pump in series. Mention may be made of a river water supply system, a hi-fi system, an office sprinkler system used for fire-extinguishing, an uninterrupted power supply (UPS) in an alternate current power supply system, TV/video system, and many more.

The novelty of this paper is that the method developed here is capable of accommodating any number of subsystems. Moreover, no fixed form of component life distribution has been assumed here. The proposed method is simple to apply and produces an explicit form for getting an optimal solution.

The paper is organized as follows: Section 2 discusses the model with necessary notation used in the paper. Section 3 derives the main theorem, which is needed to determine the rule for optimal allocation of redundancy. Section 4 presents a numerical example to illustrate the method with a sensitivity analysis. Section 5 concludes the paper with a discussion.

2 THE MODEL

An n-component coherent system is considered here, which can be decomposed into k non-overlapping subsystems in such a way that the system fails with the failure of any of the subsystems, while a subsystem fails when all of its constiuent components fails, and no two subsystems share any component. In a coherent system every component is relevant and the system is monotone [7], i.e., the system performance improves with the improvement of any component or a subset of components. Here the lifetimes of the components are assumed to be independently distributed among themselves and independent of the lifetimes of the redundant components.

Notation:

Following notation are used in this paper:

Y_j : random life of j^{th} component, $j = 1, 2, \ldots, n$

M_i : i^{th} subsystem, $i = 1, 2, \ldots, k$

n_i : size of i^{th} subsystem M_i, $i = 1, 2, \ldots, k$

p_i : reliability of the components belonging to the i^{th} subsystem, $i = 1, 2, \ldots, k$,

x_i : number of active redundant components to be attached to the components of i^{th} subsystem in order to minimize the total cost of using redundancy under reliability constraint, $i = 1, 2, \ldots, k$,

$\mathbf{x} \equiv (x_1, x_2 ..., x_k)$: vector of the redundancy numbers

r_i : reliability of the redundant component that is to be added to the i^{th} subsystem

R : specified target level of system reliability

c_i : cost of adding a redundant component to the components of i^{th} subsystem, $i = 1, 2,\ldots, k$,

$C(\mathbf{x}) \equiv C(x_1, x_2 ..., x_k)$: cost function for adding rdundancy

$q_F^{(i)}$: unreliability of the i^{th} subsystem, M_i.

The system reliability $R(t)$ at time t is given by

$$R(t) = P(T > t) = E[\prod_{i=1}^{k}\{\coprod_{l \in M_i} u_l(t)\}], \qquad (1)$$

where T is the system life, state variable $u_l(t)$ is 0 if l^{th} component is in failing state, and 1, if l^{th} component is in functioning state at time t with $E(u_l(t)) = p_l(t)$.

Thus (1) shows that the system reliability can be expressed in terms of its subsystem reliabilities that involves component reliabilities $p_l(t)$'s. From now on, Henceforth the time variable t will be suppressed for notational simplicity. It will not affect the optimal solution anyway, because the decision is to be made for the design stage only.

The reliabilities of the components belonging to a same subsystem can reasonably be assumed to be same, and the reliabilities of the redundant components that are to be added to a same subsystem can also assumed to be same. Thus the reliabilities of all components belonging to the i^{th} subsystem are p_i, and the reliabilities of redundant components to be added to the i^{th} subsystem are r_i (p_i's and r_i's may or may not be the same), for all i =1, 2, …, k. Then, from (1), the reliability of the augmented system can be written as

$$R(\mathbf{x}) = \prod_{i=1}^{k}[1 - \{\prod_{j \in M_i}(1 - p_j)\} \times (1 - r_i)^{x_i}]$$

$$= \prod_{i=1}^{k}[1 - (1 - p_i)^{n_i} \times (1 - r_i)^{x_i}], \qquad (2)$$

since here the reliability of i^{th} subsystem, M_i, having n_i components is

$$\{\prod_{j \in M_i}(1 - p_j)\} \times (1 - r_i)^{x_i} = (1 - p_i)^{n_i} \times (1 - r_i)^{x_i}.$$

The present work finds out a rule for deciding the optimal redundancy number that should be added to different subsystems so that the total cost for adding redundancy is minimized, as well as we can ensure a minimum level of system reliability, which we call a reliability target. The results have been derived under the assumption that the components belonging to the same subsystem have same reliability. The reliabilities of all redundant components that are to be added to the same subsystem are also assumed to be same. Sometimes it may further be reasonable to assume the reliability of the redundant components to be same as that of the components belonging to the respective subsystem, to which the redundant components are to be added.

The number of redundant components, $\mathbf{x} = (x_1, x_2 ..., x_k)$, should be so determined that $R(\mathbf{x}) \equiv R(x_1, x_2 ..., x_k) \geq R$ and $C(\mathbf{x}) \equiv C(x_1, x_2 ..., x_k)$ is minimized.

Here our objective is to determine the optimal values of decision variables, $x_1, x_2 ..., x_k$ that minimize the cost function $C(\mathbf{x}) \equiv C(x_1, x_2 ..., x_k)$, a linear objective function, subject to a reliability target R, where, by (2), the system reliability is as follows:

$$R(\mathbf{x}) = \prod_{i=1}^{k}\{1 - (1 - p_i)^{n_i} \times (1 - r_i)^{x_i}\}.$$

The problem is to minimize $C = C(\mathbf{x}) = \sum_{i=1}^{k} c_i x_i$

subject to $R(\mathbf{x}) \geq R$,

where $x_i \geq 0$, $c_i \geq 0$, for all i = 1, 2, …, k.

The problem can, alternatively, be stated in an equivalent form of a maximization problem as follows:

maximize $-C(\mathbf{x}) = -\sum_{i=1}^{k} c_i x_i$

subject to $-\log_e R(\mathbf{x}) \leq -\log_e R$.

Next section determines the optimal solution to the above problem.

3 MINIMIZING COST UNDER RELIABILITY CONSTRAINT

Let us first prove the following lemma which will be required to obtain the optimal solution $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_k^*)$ that minimizes $C(\mathbf{x})$.

Lemma 1. $\log_e R(\mathbf{x})$ *is a concave function.*

Proof. The diagonal elements of the Hessian matrix of the function $\log_e R(\mathbf{x})$ are:

$d_{ii} = \partial^2 \log_e R(\mathbf{x}) / \partial x_i^2$, $i = 1, 2, ..., k$,

and the off-diagonal elements are:

$d_{ij} = d_{ji} = \partial^2 \log_e R(\mathbf{x}) / \partial x_i \partial x_j$, $i \neq j, i, j = 1, 2, ..., k$.

Here

$$\frac{\partial \log_e R(\mathbf{x})}{\partial x_i} = \frac{\partial \log_e[\prod_{i=1}^{k}\{1-(1-p_i)^{n_i}(1-r_i)^{x_i}\}]}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i}[\sum_{i=1}^{k}\log_e\{1-(1-p_i)^{n_i}\times(1-r_i)^{x_i}\}],$$

or, $$\frac{\partial \log_e R(\mathbf{x})}{\partial x_i} = \frac{\{q_F^{(i)}\times(1-r_i)^{x_i}\}\times\log_e(\frac{1}{1-r_i})}{1-\{q_F^{(i)}\times(1-r_i)^{x_i}\}}, \quad (3)$$

where $q_F^{(i)} = \prod_{j\in M_i}(1-p_j) = (1-p_i)^{x_i}$.

From (3),

$$\frac{\partial^2 \log_e R(\mathbf{x})}{\partial x_i^2} = \frac{\partial}{\partial x_i}[\frac{q_F^{(i)}\times(1-r_i)^{x_i}\times\log_e(\frac{1}{1-r_i})}{1-\{q_F^{(i)}\times(1-r_i)^{x_i}\}}]$$

$$= \log_e(\tfrac{1}{1-r_i})\times\frac{q_F^{(i)}\times(1-r_i)^{x_i}\times\log_e(1-r_i)}{1-\{q_F^{(i)}\times(1-r_i)^{x_i}\}}$$

$$\times[1+\frac{q_F^{(i)}\times(1-r_i)^{x_i}}{1-\{q_F^{(i)}\times(1-r_i)^{x_i}\}}], \text{ for all } i=1,2,...,k,$$

< 0, since $\log_e(1-r_i) < 0$, and

$$\frac{\partial^2 \log_e R(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}[\frac{q_F^{(i)}\times(1-r_i)^{x_i}\times\log_e(\frac{1}{1-r_i})}{1-\{q_F^{(i)}\times(1-r_i)^{x_i}\}}] =$$

for all $i, j = 1, 2, \ldots, k$, $j \neq i$.

Here all odd-ordered minors of the Hessian matrix of $\log_e R(\mathbf{x})$ are negative and even-ordered minors are positive, and hence the matrix is negative definite, indicating concavity of the function $\log_e R(\mathbf{x})$.

Note that $-\log_e R(\mathbf{x})$ is a convex function.
Let us now consider the following Lagrange function:

$$G(\mathbf{x}) = -\sum_{i=1}^{k} c_i x_i - \lambda\{-\log_e R(\mathbf{x}) + \log_e R\}$$

$$= -[\sum_{i=1}^{k} c_i x_i - \lambda\{\log_e R(\mathbf{x}) - \log_e R\}], \quad (4)$$

where the constant λ is a positive real number, known as Lagrangian multiplier.

Note that $G(\mathbf{x})$ is a convex function, being a linear combination of two convex functions, $-\log_e R(\mathbf{x})$, by Lemma 1, and $-\sum_{i=1}^{k} c_i x_i$ (which is concave as well, being a linear function).

The next lemma finds the stationary point of $G(\mathbf{x})$ at which the Jacobian of the real-valued function $G \equiv G(\mathbf{x})$ is a null vector, i.e., $\nabla G(\mathbf{x}) = \mathbf{0}$, where

$$\nabla G(\mathbf{x}) = \left(\frac{\partial G(\mathbf{x})}{\partial x_1}, \frac{\partial G(\mathbf{x})}{\partial x_2}, ..., \frac{\partial G(\mathbf{x})}{\partial x_k}\right)'.$$

Lemma 2. *The stationary point of* $G(\mathbf{x})$, *at which* $\nabla G(\mathbf{x}) = \mathbf{0}$, *is given by*

$$x_i = \frac{1}{\log_e(1-r_i)} \times\left[\log_e\{\frac{c_i}{c_i+\lambda\log_e(\frac{1}{1-r_i})}\} - n_i\times\log_e(1-p_i)\right],$$

$i = 1, 2, ..., k.$

Proof. Here, from (4), we get

$$\frac{\partial G(\mathbf{x})}{\partial x_i} = -c_i + \lambda\times\left[\frac{(1-p_i)^{n_i}\times(1-r_i)^{x_i}\times\log_e(\frac{1}{1-r_i})}{1-\{(1-p_i)^{n_i}\times(1-r_i)^{x_i}\}}\right].$$

$\frac{\partial G(\mathbf{x})}{\partial x_i} = 0$ gives

$$x_i = \frac{1}{\log_e(1-r_i)} \times\left[\log_e\{\frac{c_i}{c_i+\lambda\log_e(\frac{1}{1-r_i})}\} - n_i\times\log_e(1-p_i)\right],$$

$$i = 1, 2, ..., k. \quad (5)$$

Here λ can be obtained from the reliability constraint $\log_e R(\mathbf{x}) = \log_e R$.
Hence the result.

Now we apply a multivariate constrained optimization technique to determine the optimal number of redundant components for all subsystems. Let $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_k^*)$ and λ^* be the solution of $(k + 1)$ equations - the k equations, as given by (5), and the reliability constraint $\log_e R(\mathbf{x}) = \log_e R$. Since $G(\mathbf{x})$ is convex, ($\mathbf{x}^*$, λ^*) minimizes $G(\mathbf{x})$. Next we will show that ($\mathbf{x}^*$, λ^*) will minimize the cost function $C(\mathbf{x})$ as well. In fact, ($\mathbf{x}^*$, λ^*) is the global minimizer of $C(\mathbf{x})$, which will be shown in the following theorem.

The sufficient condition for global minimum in the problem of minimizing a function $f(\mathbf{v}) \equiv f(v_1, v_2, ..., v_k)$ subject to $h_l(\mathrm{v}) \equiv h_l(v_1, v_2, ..., v_k) = b_l$, l = 1, 2, …, m, is given below, which will be required to prove the main theorem. The conditions are known as Kuhn-Tucker conditions [8].

If $\mathbf{v}^* = (v_1^*, v_2^*, ..., v_k^*)$ is a minimizer, then there exists a $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_l^*)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ satisfies conditions

$$\left.\begin{aligned} &\nabla f(\mathbf{v}^*) - \boldsymbol{\lambda}^* \times \nabla h(\mathbf{v}^*) = 0 \\ &\text{and } h_l(\mathbf{v}^*) = b_l, l = 1, 2, ..., m. \end{aligned}\right\} \tag{6}$$

Then the sufficient condition for global minimum is:

If $f(v_1, v_2, ..., v_k)$ *is a convex function, each* $h_l(v_1, v_2, ..., v_k)$ *is concave,* $(\mathbf{v}^*, \boldsymbol{\lambda}^*)$ *satisfies the conditions* given in (6), *and* $\lambda_l \geq 0$, l = 1, 2, …, m, *then* $\mathbf{v}^*$ *is a global minimizer of* $f(v_1, v_2, ..., v_k)$.

Now the main result, stated in Theorem 1 below, shows that ($\mathbf{x}^*$, λ^*) is the global minimizer of $C \equiv C(\mathbf{x}) = \sum_{i=1}^{k} c_i x_i$, and hence gives the optimal solution that minimizes total cost due to adding redundancy.

Here $\mathbf{v} \equiv \mathbf{x}$ (hence $\mathbf{v}^* \equiv \mathbf{x}^*$), $f(\mathbf{v}) \equiv C(\mathbf{x})$, $h_l(\mathbf{v}) \equiv \log_e R(\mathbf{x})$, m = 1 (hence l = 1), $b_l \equiv \log_e R$.

Theorem 1. ($\mathbf{x}^*$, λ^*), that minimizes $G(\mathbf{x})$, *is a global minimizer of* $C \equiv C(\mathbf{x})$.

Proof. As given in (4),

$$G(\mathbf{x}) = -\sum_{i=1}^{k} c_i x_i - \lambda\{-\log_e R(\mathbf{x}) + R\},$$

from which we find

$$\frac{\partial G(\mathbf{x})}{\partial x_i} = -\{\frac{\partial}{\partial x_i}(\sum_{i=1}^{k} c_i x_i)\} - (-\lambda \times \frac{\partial \log_e R(\mathbf{x})}{\partial x_i}).$$

Thus ($\mathbf{x}^*$, λ^*), the stationary point of $G(\mathbf{x})$, which is the solution of $\nabla G(\mathbf{x}) = \mathbf{0}$, satisfies the following conditions:

$$-\{\frac{\partial}{\partial x_i}(\sum_{i=1}^{k} c_i x_i)\} + \lambda \times \frac{\partial \log_e R(\mathbf{x})}{\partial x_i} = 0, \text{ at } (\mathbf{x}^*, \lambda^*),$$

i.e., $$-\left[\frac{\partial}{\partial x_i}(\sum_{i=1}^{k} c_i x_i)\right]_{\mathbf{x}^*} + \lambda^* \times \left[\frac{\partial L}{\partial x_i}\right]_{\mathbf{x}^*} = 0$$

and $\log_e R(\mathbf{x}) = \log_e R$.

Thus the sufficient condition for a global minimum, as given in (6), is satisfied.

Hence the result.

Using Lemmas 1, 2 and Theorem 1, the optimum solution (global minimizer) $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_k^*)$ of $\mathbf{x} = (x_1, x_2, …, x_k)$, can be obtained by solving k equations, as given by (5), subject to $\sum_{i=1}^{k} \log_e\{1-(1-p_i)^{n_i+x_i}\} = \log_e R$, so that $C(\mathbf{x})$ is minimized. Thus, in general, we get the optimal solution as

$$x_i = \frac{1}{\log_e(1-r_i)} \times \left[\log_e\{\frac{c_i}{c_i + \lambda \log_e(\frac{1}{1-r_i})}\} - n_i \times \log_e(1-p_i)\right],$$
$$i = 1, 2, ..., k. \tag{7}$$

In particular, if $r_i = p_i$, for all i = 1, 2, …, k, (7) reduces to

$$x_i = \frac{\log_e(\frac{c_i}{c_i + \lambda \log_e(\frac{1}{1-p_i})})}{\log_e(1-p_i)} - n_i,\ i = 1, 2, …, k.$$

4 A NUMERICAL EXAMPLE

Let us consider the following system, as shown in Fig. 1.

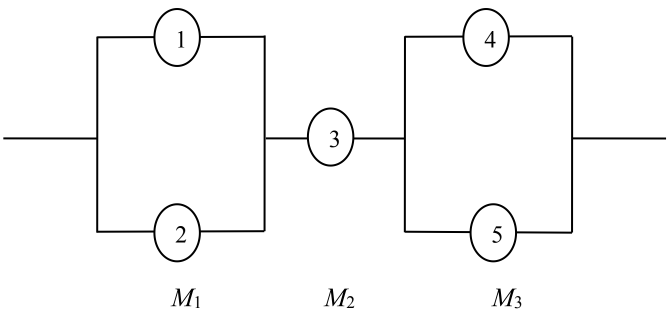

Fig. 1. A hi-fi system

The above system is decomposed into three subsystems, viz., M_1 = {1, 2}, M_2 = {3}, M_3 = {4, 5}. The reliability of each of the components belonging to subsystem 1, is p_1 = 0.9. The component reliabilities for subsystems 2 and 3 are, respectively, p_2 = 0.85 and p_3 = 0.95. Suppose, the reliability target is R = 0.9. By (1), the system reliability is 0.839396. The cost due to adding a redundant component to the subsystem 1 is $c_1 = 40$. The costs for adding a redundant component to subsystem 2 is $c_2 = 20$ and to subsystem 3 is $c_3 = 30$. The reliabilities of the redundant components to be added to the subsystems are, respectively, $r_1 = 0.9$, $r_2 = 0.85$ and $r_3 = 0.95$, which are same as the component reliabilities of respective subsystems. The number of components in the subsystems are, respectively, $n_1 = 2$, $n_2 = 1$, $n_3 = 2$. Then the solution for λ is found to be $\lambda = 226.6948$, and the number of redundant components that are to be added to different subsystems, subject to the reliability constraint, are $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, for which the minimum total cost becomes 20, and the system reliability becomes 0.9653, with a gain of 14.99936% in reliability.

Table 1 shows the optimal allocation of redundant components for different reliability targets. It reflects how sensitive the optimal solution, minimum cost and system reliability are to the change in reliability target.

Table 1. Sensitivity of optimal redundancy allocation, minimum total cost and system reliability to the reliability target

Reliability target	*Optimal allocation*			*Total cost (min)*	*Augmented system reliability*	*Gain in system reliability (%)*
	x_1	x_2	x_3			
0.85	0	1	0	20	0.9653	15.00
0.90	0	1	0	20	0.9653	15.00
0.95	0	1	0	20	0.9653	15.00
0.96	0	1	0	20	0.9653	15.00
0.97	0	2	0	40	0.9841	17.23
0.98	0	2	0	40	0.9841	17.23
0.99	1	2	0	80	0.9931	18.31
0.995	1	3	0	100	0.9960	18.66
0.999	1	18	10	700	0.9990	19.01

5 CONCLUSION AND DISCUSSION

Here a constrained cost minimizing redundancy allocation problem is solved for a complex coherent system which can be decomposed into a number of subsystems such that the entire system fails with the failure of any of the subsystems, and a subsystem fails if all of its components fail. The total cost of using redundancy has been minimized subject to a given reliability target. The method can be applied to any form of life distributions of any complex or simple coherent system. There is no restriction on the number of subsystems that constitute the whole system under consideration. A sensitivity analysis has been done to show how sensitive the optimal solution (allocation of redundant components to different subsystems), total cost and system reliability are to the given reliability target. The optimal solution, percentage gain in system reliability and total cost of using redundancy are robust within a class of values of reliability targets, but sensitive from one class to the other. The range of values within a class is wider for smaller values of reliability target, while narrower for higher values. Thus adding redundancy does not improve the system reliability much after a certain point. Hence achieving an extremely high (as high as 0.999) reliability target by adding redundancy needs a large number of redundant components, and hence becomes tremendously expensive.

Reference

[1] D. F. Morrison, The optimum allocation of spare components in systems, *Technometrics*, 3(3), 399-406 (1961).

[2] P.J. Boland, E. El-Neweihi and F. Proschan, Stochastic order for redundancy allocation in series and parallel systems, *Adv. Appl. Prob*., 24, 161-171 (1992).

[3] M. Shaked and J.G. Shanthikumar, Optimal allocation of resources to nodes of parallel and series systems, *Adv. Appl. Prob*., 24, 894-914 (1992).

[4] D.W. Coit and A.E. Smith, Reliability optimization of series-parallel systems using a genetic algorithm, *IEEE Trans. Reliab*., 45, 254-260 (1996).

[5] F. Ahmadizar and H. Soltanpanah, Reliability optimization of a series system with multiple-choice and budget constraints using an efficient ant colony approach, Expert Systems with Applications, 38, 3640-3646 (2011).

[6] N. Ruan and XL. Sun, An exact algorithm for cost minimization in series reliability systems with multiple component choices, Appl. Math. Comput., 181, 732-741 (2006).

[7] R.E. Barlow and F. Proschan, Statistical Theory of Reliability and Life Testing, To Begin with, Silver Spring, MD (1981).

[8] H.W. Kuhn and A.W. Tucker, Nonlinear programming, in Proceedings of Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, California, 481-492 (1951).