Adapted Dynamic Program to Find Shortest Path in a Network having Normal Probability Distribution Arc Length

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Abstract. We adapt a dynamic program to find the shortest path in a network having normal probability distributions as arc lengths. Two operators of sum and comparison need to be adapted for the proposed dynamic program. Convolution approach is used to sum two normal probability distributions being employed in the dynamic program. Generally, stochastic shortest path problems are treated using expected values of the arc probabilities, but in the proposed method using distributed observed past data as arc lengths, an integrated value is obtained as the shortest path length.
Keywords: Shortest path; Dynamic program; Convolution; Normal distribution

Introduction

The shortest path problem has been widely studied in the fields of operations research, computer science, and transportation engineering. There are several methods to find the shortest path from the source node to the sink node based on dynamic programming, zero-one programming and also network flows theory when the arc lengths are constant. Some of these algorithms can be found in Bazaraa (1990). Many efficient algorithms have been developed by (Bellman, 1958; Dijkstra, 1959; Dreyfus, 1969). These algorithms are called the standard shortest path algorithms. It should be mentioned that the standard shortest path algorithms only can be used to compute shortest paths in time-dependent but not stochastic networks (Orda and Rom, 1990; Kaufman et al., 1993; Ziliaskopoulos and Mahmassani, 1993; Chabini, 1997).Deo and Pang (1984) provided a taxonomy and annotation for the shortest path algorithms.

However, due to failure, maintenance or other reasons, we encountered different kinds of uncertainties in practice, and these uncertainties must be taken into account. For example, the lengths of the arcs are assumed to represent transportation time or cost rather than the geographical distances, as time or cost fluctuate with traffic or weather conditions, payload and so on, it is not practical to consider each arc as a deterministic value. In these cases, probability theory has been used to attack randomness, and many researchers have done lots of work on stochastic shortest path problem. When the arc lengths are random variables, the problem will become more difficult. Frank (1969) computed the probability that the time of the shortest path of the network is smaller than a specific value where link travel times are random variables but not time dependent. Mirchandani (1976) presented another method for obtaining the distribution function of shortest path in stochastic networks. It is not required to solve multiple integrals in this paper, but this method can only be used for the special case where arc lengths are discrete random variables. Loui (1983), Mirchandani and Soroush (1986), and Murthy and Sarkar (1996) considered the different types of cost functions to study the variations of the shortest path problem in stochastic networks. It was found that for identifying the expected shortest path the random link travel times can be replaced by their expected values, then the problem simply reduces to a deterministic shortest path problem. Therefore, the efficient standard shortest path algorithms still can be used to find the expected shortest paths in a static and stochastic network.

There are several papers about finding a path with minimum expected value or minimum variance or other criteria in stochastic networks. In these papers, the multi criteria networks are analyzed. Sigal et al. (1980), studied the probability distribution of the shortest path length when arc lengths are random variables. Martins (1984) provided set of efficient paths for bicriteria shortest path problem by dynamic programming. Current and Min (1986) developed a taxonomy and annotation for the multi-criteria networks.

The more general case of the stochastic, time-dependent shortest path problem was first studied by Hall (1986). He introduced the problem of finding least expected travel time path between two nodes in a network with travel times that are both random and time-dependent.

Fu (1998) studied the expected shortest paths in dynamic and stochastic networks in a traffic network where the link travel times are modeled as a continuous-time stochastic process. He showed that the replacement of the probability distribution for link delays by their expected values would yield sub-optimal results and prescribed a dynamic programming algorithm to solve the problem using conditional probability theory. Kaufman and Smith (1990) subsequently showed that the time-space network formulation and expected link delays could be used to solve the problem if the consistency assumption is satisfied. Fan et al. (2005) minimize expected travel time from any origin to a specific destination in a congestible network with correlated link costs. Bertsimas and Van Ryzin (1991) introduced and analyzed a model for stochastic and dynamic vehicle routing, in which a single, un-capacitated vehicle traveling at a constant velocity in a Euclidean region must serve demands whose time of arrival, location and on-site service are stochastic. (Bertsimas and Van Ryzin, 1993) extended this analysis and considered the problem of m identical vehicles with unlimited capacity.

Miller et al. (1994) have prescribed an efficient label correcting algorithm to obtain Pareto optimal paths by discretizing the probability distribution of the link delays. Psaraftis and Tsitsiklis (1993) examined shortest path problems, in which arc costs are the known functions of certain environmental variables at network nodes, and each of these variables evolves according to an independent Markov process. The vehicle can wait at a node in anticipation of more favorable arc costs. They showed that the optimal policy essentially classifies the state of the environmental variable at a node into two categories: green states for which the optimal action is to immediately traverse the arc, and red states for which the optimal action is to wait. Then they extended these concepts for the entire network by developing a dynamic programming, which solves the corresponding problem. Ji (2005) studied the shortest path problem with stochastic arc length, where according to different decision criteria three types of models were presented. In order to solve these models, a hybrid intelligent algorithm integrating stochastic simulation and genetic algorithm has been developed. Real applications of various artificial intelligence techniques in different engineering fields can be found in (Cheng et al., 2008; Taormina et al., 2012; Lin et al., 2006; Jia et al., 2008; Huang and Chau, 2008; Chau, 2007).

In this paper, due to drawbacks considered in previous stochastic shortest path problem namely, lack of validation for large scale observed data, disability in considering probability distribution, low accuracy of probabilities, we propose a novel combination of methods to find shortest path in normally distributed arc lengths network. Then, an adapted dynamic program is developed to find the optimal path in the stochastic network.

2. Problem definition and modelling

Consider a network as shown in Figure 1 consisting of a finite set of nodes and arcs of the directed acyclic network. We assume that the admissible paths are always continuous and the length of each arc is normal random variable with parameters μ and σ^2. We want to find the shortest path from the source node 1 to the sink node N using the backward dynamic programming approach.

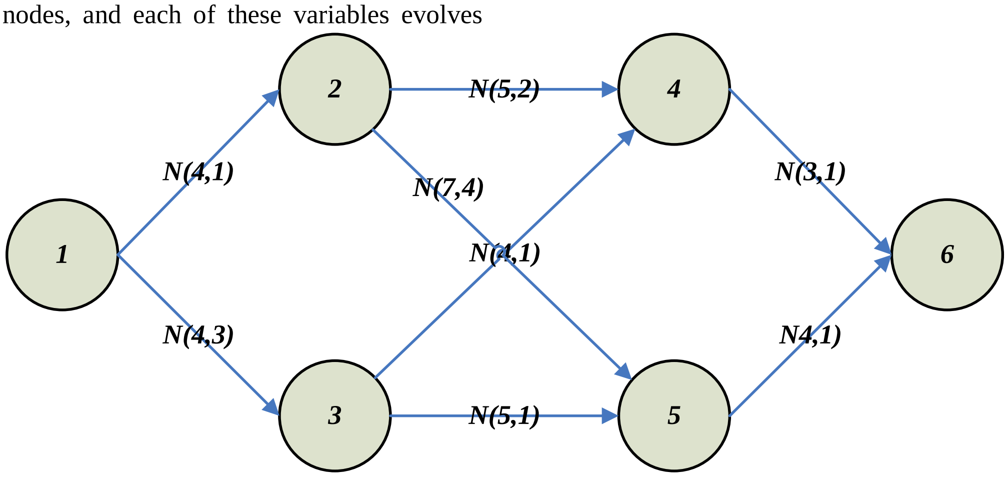

Figure 1. Acyclic network with normal distributed arcs

The optimal value function S_i can be defined by

S_i = the distribution of the shortest path from node i to node N.

Then the recurrence relation can be stated as

$$S_i = \min_{j>i} \lfloor d_{ij} + S_j \rfloor \qquad \text{For} \quad i = N-1,\dots 1 \tag{1}$$

And the boundary condition is

$$S_N = 0.$$

In this paper, we use convolution to find distribution of sum of two normal distributions in each stage in each stage. The reason is the inefficiency of the methods such as maximum likelihood estimation and moment generating function in long computational efforts and inaccurate solution results.
And for comparison in each stage we find the probability that a random variable with first distribution become smaller than another random variable with second distribution. In order to show the operation in each stage, represent the convolution and comparison between two normal density functions in subsequent section.

Definition 1: Let *X* and *Y* be two continuous random variables with density functions *f*(*x*) and *g*(*y*), respectively. Assume that both *f*(*x*) and *g*(*y*) are defined for all real numbers. Then the convolution *f *g* of *f* and *g* is the function given by

$$(f * g)(z) = \int_{-\infty}^{+\infty} f(x)\, g(z-x)\, dx$$
$$= \int_{-\infty}^{+\infty} f\,(z-y)\, g(y)\, dy$$

Theorem 1: Let *X* and *Y* be two independent random variables with density functions $f_X(x)$ and $f_Y(y)$ defined for all *x*. Then the sum *Z* = *X* +*Y* is a random variable with density function $f_Z(z)$, where f_Z is the convolution of f_X and f_Y.

$$f_z(z) = \int_{-\infty}^{+\infty} f_{X,Y}\,(x, z-x)\, dx$$
$$= \int_{-\infty}^{+\infty} f_{X,Y}\,(z-y, y)\, dy \tag{2}$$

Proof: as we knew the joint density function of independent variables is equal to the products of their density functions therefore to find density function of Z = X + Y we apply cumulative distribution function technique.

$$P(Z \le z) = P(X+Y \le z)$$
$$= \int_{-\infty}^{+\infty} P(X+Y \le z | X=x)\, f_X(x)$$
$$= \int_{-\infty}^{+\infty} P(x+y \le z)\, f_X(x) dx$$
$$= \int_{-\infty}^{+\infty} F_Y(z-x)\, f_X(x) dx$$

Now, we set partial derivative to obtain the summation density function

$$f_Z(z) = \frac{dF_Z\,(z)}{dz} = \frac{d}{dz}\left[\int_{-\infty}^{+\infty} F_Y(z-x\,)\, f_X(x) dx\right]$$
$$= \int_{-\infty}^{+\infty} \frac{dF_Y(z-x\,)}{dz} f_X(x) dx$$
$$= \int_{-\infty}^{+\infty} f_Y(z-x)\; f_X(x) dx$$

2.1. Sum of two independent normal random variables

Suppose that we have two random variables X and Y with a normal density function with parameters μ and σ^2. We represent the density function of Z = X + Y as follows

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}},$$
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{\frac{-(x-\mu_y)^2}{2\sigma_y^2}}.$$

$$\int_{-\infty}^{+\infty} f_Y(z-x) f_X(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y} e^{\frac{-(z-x-\mu_y)^2}{2\sigma_y^2}} \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2+\sigma_y^2}} \exp\left[-\frac{(z-(\mu_x+\mu_y))^2}{2\,(\sigma_x^2+\sigma_y^2)}\right] \frac{1}{\sqrt{2\pi}\frac{\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}} \exp\left[-\frac{\left(x-\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_x^2+\sigma_y^2}\right)^2}{2\left(\frac{\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2 + \sigma_y^2}} \exp\left[-\frac{(z-(\mu_x+\mu_y))^2}{2(\sigma_x^2+\sigma_y^2)}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}} \exp\left[-\frac{\left(x-\frac{\sigma_x^2(z-\mu_y)+\sigma_y^2\mu_x}{\sigma_x^2+\sigma_y^2}\right)^2}{2\left(\frac{\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2}\right] dx$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2 + \sigma_y^2}} \exp\left[-\frac{(z-(\mu_x+\mu_y))^2}{2(\sigma_x^2+\sigma_y^2)}\right] \tag{3}$$

Result 1:if X_1, X_2,…, X_n are independent normal random variables with (μ_1,σ_1^2), (μ_2,σ_2^2),…, (μ_n,σ_n^2) , then $Y = \sum_{i=1}^{n} X_i$ follows normal distribution with $(\sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i^2)$.

2.2. Finding minimum density function

Now we illustrate the method that we use to find minimum between two normal random variables. In order to find the minimum random variable we compute the probability that the first random variable X_1 with normal density function with (μ_1,σ_1^2) became smaller than the second random variable X_2 with normal density function with (μ_2,σ_2^2) with considering result 1 we have

$$P(X_1 < X_2) = P(X_1 - X_2 < 0) = P\left(Z < \frac{0-(\mu_1-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) = \varphi\left(\frac{\mu_2-\mu_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) \tag{4}$$

3. Numerical example

Consider the network depicted in Figure 1. We want to obtain the shortest path from node 1 to node 6 where arcs have normal distribution.

Boundary condition is

$$S_6 = 0.$$

Using the recurrence relation (1) we have

$$S_5 = N(4,1), S_4 = N(3,1)$$

For each arc doesn't exist in network we replace infinity for d_{ij}, so operations of S_3 can be stated as

$$S_3 = min\begin{bmatrix} N(4,1) + S_4 \\ N(5,1) + S_5 \end{bmatrix} = min\begin{bmatrix} N(4,1) + N(3,1) \\ N(5,1) + N(4,1) \end{bmatrix}$$

Using result 1 we have

$$S_3 = min\begin{bmatrix} N(7,2) \\ N(9,2) \end{bmatrix}$$

We find the minimum value between two normal random variables by using formula (4) as follows

$$P(X_1 < X_2) = P(X_1 - X_2 < 0) = P\left(Z < \frac{\mu_2-\mu_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\right) = P\left(Z < \frac{9-7}{\sqrt{2+2}}\right) = \varphi\left(\frac{2}{\sqrt{4}}\right) = \varphi(1) = 0.8413$$

So the first density function is minimal,

$$S_3 = N(7,2)$$

We illustrate the operation of node 2 as follows

$$S_2 = min\begin{bmatrix} N(5,2) + S_4 \\ N(7,4) + S_5 \end{bmatrix} = min\begin{bmatrix} N(5,2) + N(3,1) \\ N(7,4) + N(4,1) \end{bmatrix} = min\begin{bmatrix} N(8,3) \\ N(11,5) \end{bmatrix}$$

To find minimum density, we make use of the following probability,

$$P(X_1 < X_2) = \varphi\left(\frac{11-8}{\sqrt{8}}\right) = \varphi\left(\frac{3}{\sqrt{8}}\right) = \varphi\left(\frac{3}{4\sqrt{2}}\right) = 0.8556$$

So with probability 0.8556 we choose first density function as minimum density function.

$$S_2 = N(8,3)$$

Now we do operations for S_1 to find the shortest path in network

$$S_1 = min\begin{bmatrix} N(4,1) + S_2 \\ N(4,3) + S_3 \end{bmatrix} = min\begin{bmatrix} N(4,1) + N(5,2) + N(3,1) \\ N(4,3) + N(4,1) + N(3,1) \end{bmatrix} = min\begin{bmatrix} N(12,4) \\ N(11,5) \end{bmatrix}$$

$$P(X_1 < X_2) = \varphi\left(\frac{11-12}{\sqrt{9}}\right) = \varphi\left(\frac{-1}{3}\right) = 0.3694$$

$$S_1 = N(11,5)$$

Therefore the shortest path in the network is 1-3-4-6 and has normal distribution with mean 11 and variance 5 having the probability of 0.6306.

4. Conclusions

This paper proposed an adapted dynamic program for determining the shortest path in a normal probability distribution arc length network. Since the definite values of the dynamic program were turned into normal random variables, two modifications were performed on sum and comparison operators. Convolution technique was employed for summing two normal probability distributions. Numerical example via a six node network showed the performance of the proposed methodology for the shortest path. In future the problem can be solved by different methods like maximum likelihood estimation (MLE) or using Bayesian approach. We can modify the problem by changing arcs distributions to exponential or gamma density functions or mixing them in the network.

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